Analysis IV

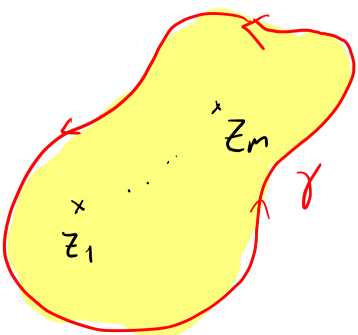
(for EL, GM, MX)

Week 7 - 31.03.25

Recall from last week:

Residue theorem:

Let γ be a simple, closed, counterclockwise oriented curve and let $z_1, \dots, z_m$ be points in the interior of γ.



If $f: \text{int}(\gamma)/\{z_1, \dots, z_m\} \to \mathbb{C}$ is holomorphic, then

$$\int_\gamma f(z)\,dz = 2\pi i \left(\text{Res}_{z_1}(f) + \dots + \text{Res}_{z_m}(f)\right).$$

Residue: The coefficient c_{-1} of the Laurent series of f at that point

$$f(z) = \dots + \frac{c_{-2}}{(z-z_0)^2} + \frac{c_{-1}}{(z-z_0)} + c_0 + c_1(z-z_0) + \dots$$

$$\Rightarrow \quad \mathrm{Res}_{z_0}(f) = c_{-1}.$$

Useful formulas: If z_0 is a pole of order 1 (simple pole), then

$$\mathrm{Res}_{z_0}(f) = \lim_{z \to z_0} \left((z-z_0) \cdot f(z) \right).$$

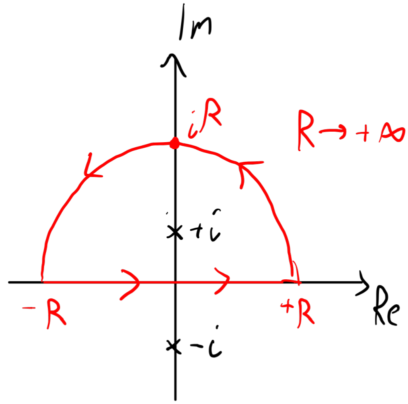
(Recall: z_0 pole of order $m \Leftrightarrow \lim_{z \to z_0} (z-z_0)^m f(z)$ exists and is non-zero).

If z_0 is a pole of order <u>at most</u> m:

$$\mathrm{Res}_{z_0}(f) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0)^m f(z) \right).$$

We also saw that we can use the residue theorem to compute certain integrals over the real line:

For instance, $\displaystyle\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx$



$\pm i$: poles of $f(z) = \frac{1}{z^2+1}$

This method works for functions $f(z)$ that, as $|z| \to +\infty$, decay faster than $1/|z|$.

Applications of the residue theorem (continued)

Let $R(x) = \frac{P(x)}{Q(x)}$, where $P(x)$, $Q(x)$ are <u>real</u> polynomials and $\boxed{\deg Q \geq \deg P + 2}$

We will also assume that $R(x)$ is defined on all of $\mathbb{R}$, i.e. that $Q(x) \neq 0$ for $x \in \mathbb{R}$.

Remarks:

- The above means that $\deg Q$ is even, since an odd-degree polynomial will always have at least one real root.

- $Q(x) = a_k x^k + a_{k-1} x^{k-1} + \ldots + a_0$ has real coefficients. So, if for some $z \in \mathbb{C}$

we have $Q(z)=0$, then by taking the complex conjugate we have $Q(\bar{z})=0$.

So the roots of Q in $\mathbb{C}$ come in pairs of complex conjugates (none on the real line).

For any $a \geqslant 0$, we are interested in computing integrals of the form

$$\int_{-\infty}^{+\infty} R(x)\cos(ax)\,dx, \quad \int_{-\infty}^{+\infty} R(x)\sin(ax)\,dx.$$

Since $e^{iax}=\cos(ax)+i\sin(ax)$, I can compute them both by calculating:

$$I=\int_{-\infty}^{+\infty} R(x)\,e^{iax}\,dx.$$

- The example $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx$ that we computed last week: Is a special case $(a=0,\ P(x)=1,\ Q(x)=1+x^2)$

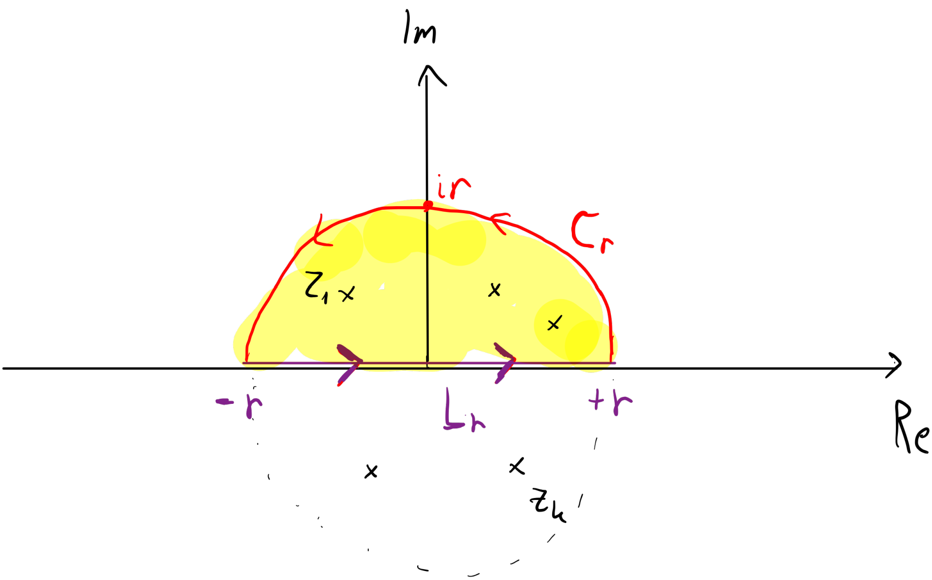
- Trigonometric integrals like the above appear often in applications:
  - Fourier transform (signal analysis):
  $$\hat{f}(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{-iat}\, dt.$$
  - Study of resonances in harmonic systems with friction.

In order to compute the integral I:

Consider the function $f(z) = R(z) \cdot e^{iaz}$.

If $z_1, z_2, \ldots, z_k$ are the zeroes of $Q(z)$,

$f: \mathbb{C} \setminus \{z_1, \ldots, z_k\} \longrightarrow \mathbb{C}$ is holomorphic.



Let $r > 0$ be large enough so that $z_1, \ldots, z_k$ are contained in the disc $B_r(0)$.

Like last week, consider the curves:

$$L_r = \{ z : \operatorname{Im}(z) = 0, \ -r \leq \operatorname{Re}(z) \leq r \}$$

parametrized by $L_r(t) = t$, $t \in [-r, r]$

$$C_r = \{ z : \ z = r \cdot e^{i\theta}, \ \theta \in [0, \pi] \}$$

(half-circle in the upper half-plane).

And $\Gamma_r = L_r \cup C_r$ (simple closed curve).

$$\int_{\Gamma_r} f(z)\, dz = \int_{L_r} f(z)\, dz + \int_{C_r} f(z)\, dz$$

$$= \int_{-r}^{+r} R(x) e^{iax}\, dx + \int_{C_r} f(z)\, dz$$

As $r \to +\infty$:

- I can compute $\int_{\Gamma_r} f(z)\,dz$ using the residue theorem:

  If $z_1, \ldots, z_m$ are the zeroes of Q in the upper half plane (these are half of the total zeroes), then

$$\int_{\Gamma_r} f(z)\,dz = 2\pi i \left(\mathrm{Res}_{z_1}(f) + \cdots + \mathrm{Res}_{z_m}(f) \right).$$

- $\displaystyle\int_{-r}^{r} R(x)\, e^{iax}\, dx \xrightarrow{r \to +\infty} I.$

- For $\int_{C_r} f(x)\,dx$, we will show that it goes to 0 as $r \to +\infty$.

For this, we will use the following facts about $f(z) = R(z) \cdot e^{iaz}$:

- For $\mathrm{Im}(z) \geqslant 0$ and $a \geqslant 0$, $|e^{iaz}| \leq 1$: Split $z = x + yi$ to calculate $|e^{iaz}| = |e^{iax - ay}|$ $= e^{-ay} \leq 1$.

  (Note: $|e^{iaz}|$ is not bounded on all of $\mathbb{C}$).

- $|R(z)|$ decays to 0 as $|z| \to +\infty$.

In particular:

$$\left| \int_{C_r} f(z)\, dz \right| = \left| \int_0^{\pi} f(r \cdot e^{i\theta}) \cdot i r e^{i\theta}\, d\theta \right|$$

$$\underset{\text{(triangle inequality)}}{\leq} \int_0^{\pi} |f(re^{i\theta})| \cdot |rie^{i\theta}| \, d\theta$$

$$= \int_0^{\pi} |f(re^{i\theta})| \cdot r \, d\theta$$

$$= \int_0^{\pi} \frac{|P(re^{i\theta})|}{|Q(re^{i\theta})|} \cdot |e^{iae^{i\theta}}| \cdot r \, d\theta \quad 1$$

As we showed above: $|e^{iaz}| \leq 1$ for $a \geq 0$ and $\text{Im}(z) \geq 0$ (so works for $z = e^{i\theta}$, $0 \leq \theta \leq \pi$).

And: Since $\deg Q \geq \deg P + 2$, if

$$P(z) = b_l z^l + b_{l-1} z^{l-1} + \dots$$

$$Q(z) = a_k z^k + a_{k-1} z^{k-1} + \dots \qquad k \geq l + 2$$

and $$\frac{P(z)}{Q(z)} = \frac{b_l z^l + b_{l-1} z^{l-1} + \dots}{a_k z^k + a_{k-1} z^{k-1} + \dots}$$

$$= \frac{1}{z^{k-l}} \cdot \underbrace{\frac{b_l + \frac{b_{l-1}}{z} + \dots}{a_k + \frac{a_{k-1}}{z} + \dots}}_{\text{This goes to } \frac{b_l}{a_k} \text{ as } |z| \to +\infty}$$

So $$\left| \frac{P(r \cdot e^{i\theta})}{Q(r \cdot e^{i\theta})} \right| \leq \frac{C}{r^{k-l}} \text{ as } r \to +\infty$$

So 1 $$= \int_0^{\pi} \left| \frac{P(r \cdot e^{i\theta})}{Q(r \cdot e^{i\theta})} \right| \cdot \left| e^{iae^{i\theta}} \right| \cdot r \, d\theta$$

$$\leq \int_0^{\pi} \frac{1}{r^{k-l}} \cdot \left| \frac{b_l + \frac{b_{l-1}}{r \cdot e^{i\theta}} + \dots}{a_k + \frac{a_{k-1}}{r \cdot e^{i\theta}} + \dots} \right| \cdot r \, d\theta$$

$$\longrightarrow 0 \quad \text{as} \quad r \to +\infty$$

(recall that $k - l \geq 2$).

So $\displaystyle\int_{C_r} f(z)\, dz \xrightarrow{r \to +\infty} 0.$

Thus: $\displaystyle\int_{-\infty}^{+\infty} R(x)\, e^{iax}\, dx = 2\pi i \left(\mathrm{Res}_{z_1}(f) + \dots + \mathrm{Res}_{z_m}(f) \right)$

with

$f(z) = R(z) \cdot e^{iaz}$, $\quad z_i$'s: zeroes of Q in the upper half plane.

Example: Compute the integral

$$A = \int_{-\infty}^{+\infty} \frac{x^2}{16 + x^4} \cos(x)\, dx$$

Previous calculation with $a = 1$,

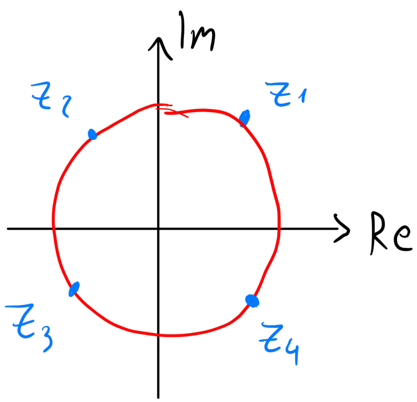
$$P(x) = x^2, \quad Q(x) = 16 + x^4,$$

$$A = \operatorname{Re}\left(\int_{-\infty}^{+\infty} \frac{x^2}{16+x^4} \cdot e^{ix}\, dx \right).$$

Roots of $Q(x)$:

$$z^4 = -16 = 16 e^{i\pi}$$

$$z = r \cdot e^{i\theta} \implies r = 2, \quad \theta = \frac{\pi}{4} + k\frac{\pi}{2},$$



$$k = 0, \dots, 3$$

$$z_1 = \sqrt{2} + \sqrt{2}i$$

$$z_2 = -\sqrt{2} + \sqrt{2}i$$

$$z_3 = -\sqrt{2} - \sqrt{2}i$$

$$z_4 = \sqrt{2} - \sqrt{2}i$$

And $Q(z) = (z-z_1)(z-z_2)(z-z_3)(z-z_4)$

(simple roots)

So $R(z) = \frac{P(z)}{Q(z)} = \frac{z^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$

Each z_i: Pole of order 1.

Only relevant for the integral: the ones with $\operatorname{Im}(\cdot) \geq 0$, so z_1, z_2.

$f(z) = R(z) \cdot e^{iz}$,

$\operatorname{Res}_{z_1}(f) = \lim_{z \to z_1} \left((z-z_1) f(z) \right) =$

$$= \frac{z_1^2 e^{iz_1}}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}$$

$$= \frac{2(1+i)^2 e^{\sqrt{2}i(1+i)}}{2\sqrt{2}\cdot 2\sqrt{2}(1+i)\cdot 2\sqrt{2}i} = \frac{e^{-\sqrt{2}}\cdot e^{\sqrt{2}i}}{4\sqrt{8}}\cdot\frac{1+i}{i}$$

$$\mathrm{Res}_{z_2}(f) = \lim_{z\to z_2}\left((z-z_2)f(z)\right)$$

$$= \frac{z_2^2 e^{iz_2}}{(z_2-z_1)(z_2-z_3)(z_2-z_4)}$$

$$= \frac{2(-1+i)^2 e^{\sqrt{2}i(-1+i)}}{-2\sqrt{2}\cdot 2\sqrt{2}i\cdot(2\sqrt{2})(-1+i)} = \frac{e^{-\sqrt{2}}\cdot e^{-\sqrt{2}i}}{4\sqrt{8}}\cdot\frac{1-i}{i}$$

So: $$\int_{-\infty}^{+\infty} R(x)e^{ix}\,dx = 2\pi i\left(\mathrm{Res}_{z_1}(f)+\mathrm{Res}_{z_2}(f)\right)$$

$$= 2\pi e^{-\sqrt{2}}\left(e^{\sqrt{2}i}+ie^{\sqrt{2}i}+e^{-\sqrt{2}i}-ie^{-\sqrt{2}i}\right)$$

$$= 2\pi e^{-\sqrt{2}}\left(2\cos\sqrt{2}-2\sin\sqrt{2}\right)$$

A: $\mathrm{Re}(\ldots) = 4\pi e^{-\sqrt{2}}(\cos\sqrt{2} - \sin\sqrt{2})$.

Example: Integrals of rational functions of trigonometric functions.

Suppose that $P(x,y)$, $Q(x,y)$ are polynomials, so that $Q(x,y)$ is non-zero on the unit circle (i.e. for $x^2+y^2=1$).

We want to compute

$$\int_0^{2\pi} \frac{P(\cos t, \sin t)}{Q(\cos t, \sin t)}\, dt.$$

We will re-express it as a complex integral over the unit circle.

Recall that:

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \quad \sin t = \frac{e^{it} - e^{-it}}{2i}$$

So: $$\int_0^{2\pi} \frac{P(\cos t, \sin t)}{Q(\cos t, \sin t)}\, dt = \int_0^{2\pi} \frac{P\left(\frac{e^{it}+e^{-it}}{2}, \frac{e^{it}-e^{-it}}{2i}\right)}{Q\left(\frac{e^{it}+e^{-it}}{2}, \frac{e^{it}-e^{-it}}{2i}\right)}\, dt$$

$$= \int_0^{2\pi} \frac{P(\ldots)}{Q(\ldots)} \cdot \frac{1}{ie^{it}} \cdot ie^{it}\, dt$$

If we set $$f(z) = \frac{P\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right)}{Q\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right)} \cdot \frac{1}{iz}$$

then the integral is equal to

$$\int_0^{2\pi} f(e^{it})\, i e^{it}\, dt = \int_\gamma f(z)\, dz$$

$$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}, \quad \gamma(t) = e^{it}.$$

(parametrization of the unit circle)

Since P, Q are polynomials: f is holomorphic on $\mathbb{C} \setminus \{z_1, \ldots, z_\ell\}$, where the z_i's are the points where the total denominator of $\frac{P(\ldots)}{Q(\ldots)}$ vanishes.

(A function which is holomorphic except for isolated points of singularity is called meromorphic).

If $z_1, \dots, z_k$ are those points in the interior of γ; By the residue theorem

$$\boxed{\int_0^{2\pi} \frac{P(\cos t, \sin t)}{Q(\cos t, \sin t)}\, dt = 2\pi i \left(\operatorname{Res}_{z_1}(f) + \dots + \operatorname{Res}_{z_k}(f) \right).}$$

Example (special case)

Compute $\displaystyle\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta$

Using the idea from the previous calculation:

Here: $P(x,y) = 1$, $Q(x,y) = 2+x$

So $$f(z) = \frac{1}{iz} \cdot \frac{P\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right)}{Q\left(\frac{z+1/z}{2}, \frac{z-1/z}{2i}\right)}$$

$$= \frac{1}{iz} \cdot \frac{1}{2+\frac{z+1/z}{2}}$$

$$= \frac{2}{i} \cdot \frac{1}{z^2+4z+1}$$

Singular points: $z^2+4z+1=0$

$\Rightarrow \quad z_1 = -2+\sqrt{3}, \quad z_2 = -2-\sqrt{3}$

Only z_1 is inside the unit circle.

And $f(z) = \frac{2}{i} \cdot \frac{1}{(z-z_1)(z-z_2)}$, so

pole of order 1 at z_1.

$$\mathrm{Res}_{z_1}(f) = \lim_{z \to z_1} \left((z - z_1) \cdot f(z) \right)$$

$$= \frac{2}{i} \cdot \frac{1}{z_1 - z_2} = \frac{1}{\sqrt{3}\, i}$$

So

$$\int_0^{2\pi} \frac{1}{2 + \cos\theta} \, d\theta = 2\pi i \; \mathrm{Res}_{z_1}(f)$$

$$= \frac{2\pi}{\sqrt{3}}$$

(Sanity check: It has to be a real number in the end!)